Research issues on punch – circular plate elastic contacts

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Abstract: Circular plates enter into category of finite dimension bodies. Solutions regarding the displacements and stresses which appear at the contact involving finite dimension bodies and also numerical and analytical results regarding some plate were reviewed in this study. For thin bodies and small transversal dimension bodies Diaconescu and Glovnea [1],[2] advanced analytical solutions solving a Boussinesq type problem for a elastic layer and a Boussinesq type problem for a cylindrical body. The obtained results satisfy all elasticity requirements so they are the searched ones. Experimental set-ups build by Glovnea [3] and Ovcharenko [4], used to investigate contact problems via laser profilometry or direct optical methods were also reviewed.

Keywords: Boussinesq, layer, body, thin, transversal dimension.

1. Introduction

Usually bodies in contact have much larger dimensions than the contact area so they can be assimilated with elastic half-spaces and the theory of half-space can be applied. There are some situations when the half-space theory cannot be applied, namely when bodies in contact have small, finite thickness or when they have small, finite transversal dimensions. This is also the case of circular plates. For these kind of situations Diaconescu and Glovnea [1] by using as a starting point an idea from Timoshenko and Goodier [5] solved two Boussinesq type of problems for finite dimension bodies. Using the results obtained by solving these problems, the integral interference equation is obtained easily by applying the superposition principle and then the contact parameters result numerically or analytically, as in [6].

2. A Boussinesq type problem for the elastic layer

Because of it’s utility, the Boussinesq’s problem is a basic tool for solving elastic contact problems. Diaconescu and Glovnea [1] reached an analytical solution for Boussinesq’s problem applied to a elastic layer subjected to compression between two equal, collinear, point forces applied normally to layer’s bounding planes. The solution consists in adding supplementary displacements to Boussinesq’s displacements from half-space theory.

\[
\begin{align*}
    u_b(r,z) &= \frac{F}{4\pi G} \left[ \frac{1}{\rho^3} - (1 - 2\nu) \frac{r}{\rho(\rho + z)} \right] \\
    w_b(r,z) &= \frac{F}{4\pi G} \left[ \frac{z^2}{\rho^3} + 2(1 - \nu) \frac{1}{\rho} \right] \\
    \sigma_{b,1}(r,z) &= -\frac{F}{2\pi} \left[ \frac{3\nu^2 z}{\rho^3} - \frac{1 - 2\nu}{\rho(\rho + z)} \right] \\
    \sigma_{b,2}(r,z) &= -\frac{F}{2\pi (1 - 2\nu)} \left[ \frac{z}{\rho^3} - \frac{1}{\rho(\rho + z)} \right]
\end{align*}
\]
\[
\sigma_{zz}(r,z) = -\frac{3F z^3}{2\pi \rho^3}
\]
\[
\tau_{zr}(r,z) = -\frac{3F r z^2}{2\pi \rho^3}
\]
where \(\rho^2 = r^2 + z^2\), \(\nu\) and \(G\) are the Poisson's ratio and shear modulus of the involved material, respectively.

An elastic layer of constant thickness \(2t\) is loaded by two normal opposed forces as shown in Figure 1. Because of the geometric and loading symmetry Diaconescu and Glovnea modeled the problem by a layer of half original layer thickness \(t\) supported without friction by a rigid flat substrate and loaded by a normal force \(F\) as we can see in Fig. 2.

The boundary conditions of this model are:

\[
\sigma_z = 0 \quad \text{and} \quad \tau_{zr} = 0 \quad \text{when} \quad z = 0 \tag{2}
\]
\[
w = 0 \quad \text{and} \quad \tau_{zr} = 0 \quad \text{when} \quad z = t \tag{3}
\]

For finding the displacements and stresses in the layer, Diaconescu and Glovnea started from the original displacements of the Boussinesq's problem for a half-space, to these displacements they added some supplementary ones so the boundary conditions be fulfilled. The total displacements from the layer will be:

\[
u = u_g + u_s, \quad w = w_g + w_s \tag{4}
\]

Because of the missing body forces, supplementary displacements must be biharmonic:

\[
\Delta \Delta u_s = 0; \quad \Delta \Delta w_s = 0 \tag{5}
\]

where \(\Delta\) is Laplace operator.

The force balance equation in a transversal plane requires:

\[
\int_0^\infty \sigma_{zz} r dr = 0 \tag{6}
\]

where \(\sigma_{zz}\) is the supplementary stress resulted from supplementary displacements \(u_s\) and \(w_s\).

Finally the following asymptotic behavior must be fulfilled:

\[
u = 0 \quad \text{and} \quad w = 0 \quad \text{when} \quad t \to \infty \tag{7}
\]

The supplementary displacement \(w_s\) must fulfill first of condition (3), after calculus results:

\[
w_s(r,z) = -\frac{F}{4\pi G} \left[ \frac{r^2}{(r^2 + t^2)^2} + \frac{1}{2(1-\nu) \sqrt{r^2 + t^2}} \right] \tag{9}
\]

Then, simultaneous fulfillment of conditions for \(\tau_{zr}\) of (2) and (3) leads to:

\[
\tilde{u}_s(r,z) = \frac{F}{4\pi G} \left[ \frac{2(1-\nu) r z}{(r^2 + t^2)^2} + \frac{3r t^2 + 6rt^2}{(r^2 + t^2)^{3/2}} \right] \tag{10}
\]

The integration of equation (10) under the first condition of (2) yields following supplementary displacement \(u_s\):

\[
u_s(r,z) = \frac{F}{4\pi G} \left[ \frac{2(1-\nu) r z}{(r^2 + t^2)^2} + \frac{1-2\nu}{r} + \frac{3r t^2 + 3rt^2}{(r^2 + t^2)^{3/2}} \right] \tag{11}
\]

Equations (9) and (11) represent the proposed supplementary displacements resulting from the finite layer thickness.

The proposed solution must fulfill all the elasticity requirements. After checking, it satisfies the boundary conditions, biharmonicity, force balance equation, asymptotic behavior and according to the principle of uniqueness of solutions in elastostatics this is the required solution.

Supplementary stresses yield from Hook's law, they added to the Boussinesq's stresses for half-space give the total stress state in the layer. Normal displacements in the layer bounding plane are:

\[
w_n(r) = \frac{2(1-\nu)}{4\pi G} \left[ \frac{1}{r} - \frac{1}{\sqrt{r^2 + t^2}} \right] \tag{12}
\]

The integration of equation (12) under the first condition of (2) yields following supplementary displacement \(u_s\):

\[
u = u_g + u_s, \quad w = w_g + w_s \tag{4}
\]

Eventually the supplementary displacement \(w_s\) must fulfill first of condition (3), after calculus results:

\[
w_s(r,z) = -\frac{F}{4\pi G} \left[ \frac{1}{r^2} \frac{1}{\sqrt{r^2 + t^2}} \right] \tag{9}
\]

Then, simultaneous fulfillment of conditions for \(\tau_{zr}\) of (2) and (3) leads to:

\[
\tilde{u}_s(r,z) = \frac{F}{4\pi G} \left[ \frac{2(1-\nu) r z}{(r^2 + t^2)^2} + \frac{3r t^2 + 6rt^2}{(r^2 + t^2)^{3/2}} \right] \tag{10}
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Supplementary stresses yield from Hook's law, they added to the Boussinesq's stresses for half-space give the total stress state in the layer. Normal displacements in the layer bounding plane are:
Knowing the stress and the displacement state from the layer, the integral interference equation of an elastic contact which involves an elastic layer can be found by applying the superposition principle. For example for the contact between an elastic punch and an elastic layer, by applying the superposition principle to the equation (12) yields the displacement \( w_0 \), the displacement \( w_1 \) is known from theory and the integral interference equation can be written:

\[
\begin{align*}
\frac{\eta}{\pi} \int \frac{p(u,v)}{\sqrt{(x-u)^2 + (y-v)^2}} \, du dv \\
= \delta - z(x,y)
\end{align*}
\] (13)

where \( \eta = (1-v^2)/E + (1-v_1^2)/E_i \) and \( \lambda = (1-v^2)/E \eta \). This equation is solved numerically by finding the major half axis. The rest of contact parameters yield from simple formulas [6]. It’s been found that contact radius decreases and the maximum pressure increases as the layer thickness decreases.

3. A Boussinesq type problem for cylindrical bodies

As we saw, regarding finite thickness bodies in contact, a Boussinesq solution was advanced to determine the stress and displacement state in them. Diaconescu and Glovnea [2] also advanced a solution for the effect of contacting body finite transversal dimension upon the elastic contact.

An apparent convenient procedure consisted in applying the results established in the fracture mechanics for a cylindrical crack. Unfortunately, these results are valid for shallow cracks, having depths bellow 0.04 of contact diameter [7], whereas in this case the depth tends to infinity, changing the problem. So the effect was classically investigated by advancing a Boussinesq type of solution for cylindrical bodies axially loaded by a point force.

Let a cylindrical elastic body having an infinite length, loaded by a central axial point force \( F \), as it’s shown in Fig. 3.

![Fig. 3. Boussinesq problem for cylindrical body, [2].](image)

To find the displacements and stresses generated into the body, the principle from the Boussinesq type of problem for elastic layers is applied.

Supplementary stresses are added to the Boussinesq’s stresses from half-space in cylindrical coordinates so the following boundary conditions are fulfilled:

\[
\begin{align*}
\sigma_z &= \sigma_{\theta \theta} + \sigma_{r z} = 0; \tau_{\theta z} = \tau_{\phi z} = 0 \quad \text{for} \quad z = 0 \\
\tau_{rr} &= \tau_{rr} = 0; \sigma_r = \sigma_{\theta \theta} + \sigma_{zz} = 0 \quad \text{for} \quad r = R
\end{align*}
\] (14)

Also the force balance equation at any depth \( z \) must be fulfilled:

\[
2\pi \int_0^R \sigma_z(r,z) r dr = -F
\] (15)

Following supplementary stress components result:

\[
\sigma_z(r,z) = \frac{F}{2\pi} \left[ \frac{3r^2 z}{(R^2 + z^2)^{3/2}} - \frac{1 - 2\nu}{\sqrt{R^2 + z^2} (z + \sqrt{R^2 + z^2})} \right]
\] (16)

\[
\sigma_r(r,z) = \frac{F}{2\pi} \left[ \frac{15R^2 r^2 z}{(R^2 + z^2)^{3/2}} - \frac{1 - 2\nu}{\sqrt{R^2 + z^2} (z + \sqrt{R^2 + z^2})} \right]
\]
These stresses satisfy the Cauchy equilibrium equations in cylindrical coordinates. The asymptotic behavior is also satisfied because when R tends to infinite all supplementary stresses and displacements vanish and Boussinesq’s solution for half-space holds. By satisfying these conditions it results that the proposed solution is correct and obviously the one searched for.

The supplementary stress \( \sigma_{zs} \) doesn’t depend on radius \( r \), this means that is uniform all over the cross-section. Its amplitude increases with \( z \) from zero tending to a constant compression stress \(-F/\pi R^3\). As the Boussinesq’s axial stress:

\[
\sigma_{zh}(r,z) = -\frac{3F}{2\pi \left(R^2 + z^2\right)^{3/2}}
\]

vanishes at high values of ordinate \( z \), the stress state becomes a uniaxial compression, in agreement with the Saint Vénant elasticity hypothesis.

When the same cylinder is subjected to a centric half-ellipsoidal pressure, authors found that the elastic state is the sum of Hertz and supplementary components [6].

4. Elastic contact punch-circular plate

Circular plates are bodies which have small thickness compared with their other dimension. For this cause they can not be assimilated with elastic half-spaces when they enter in contact problems. As we saw there were advanced some theories regarding solving the contact problems between bodies which have finite dimensions.

The equations involved in plate problems are difficult and that’s why it’s hard to obtain an exact solution. Therefore different methods were adapted for solving these kind of problems including finite element method, boundary element method, power series method, etc. Several authors solved different plate problems analytically and numerically. Chen, Cheung and Jao [8] solved plate bending problems by using a boundary element method.

The numerical results presented showed a good accuracy of this method. In Table 1 are presented the results obtained by Chen using the boundary element method for a simply supported circular plate.

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>( k_w = \frac{wEh^3}{16pa^4} )</th>
<th>( k_m = \frac{M}{4pa^2} )</th>
<th>( k_v = \frac{V}{2pa} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coordinates</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(a,0)</td>
</tr>
<tr>
<td>Analytical solution</td>
<td>0.0434766</td>
<td>0.0515625</td>
<td>0.25</td>
</tr>
<tr>
<td>Numerical solution</td>
<td>0.0434767</td>
<td>0.0515626</td>
<td>0.2500001</td>
</tr>
</tbody>
</table>

Based on it’s good accuracy and large utilization Cui, Liu, Li, Zhao and others [9] used the finite element method to analyze simply supported and clamped circular plate problems. Geometry and mesh information are shown up in Fig. 4.

![Fig. 4. Circular plate with uniform load. Geometry and mesh arrangements, [9]](image)

The obtained results were compared to other results which were also drawn by different numerical methods, the conclusion was that the presented method can produce very accurate results when the problem domain is discretized using a reasonably number of elements. Fig. 5 shows the results obtained for the central deflection in two situations a) simply supported plate and b) clamped plate.

Regarding the experimental research, a good method for contact analysis is that of laser profilometry because of it’s accurate results, repeatability and processing by using a pc.
Fig. 5. Deflection at centre for a thin circular plate with uniform load.  a) simply supported plate b) clamped plate, [9].

Glovnea [3] used this method to analyze a contact between a sapphire flat and different shaped punches. The experimental set-up is shown in Fig. 6, a) experimental set-up picture and b) schematic representation.

Fig. 6. Experimental set-up.  a) picture of test rig assembly b) schematic representation, [3].
Ovcharenko [4] used an optical direct, *in situ* technique to analyze the contact between a sapphire flat and a sphere during loading, unloading and cyclic loading-unloading. In Fig. 7 are showed the picture of the test rig assembly and the schematic representation.

In both cases the results were in good agreement with theoretical ones.

5. Conclusions

Half-space contact theory cannot be applied when either contacting bodies are thin or they possess small transversal dimensions. Diaconescu and Glovnea derived analytical solutions regarding the stress and displacement state by solving two Boussinesq type problems, one for elastic layers and one for cylindrical bodies, the solution consisted in adding supplementary displacements and stresses to Boussinesq’s ones so that elasticity conditions are fulfilled.

Knowing the solution to Boussinesq’s problem for the elastic layer, the superposition principle can be applied to yield the interference integral equation for the contact elastic punch-elastic layer, which is useful for solving these kind of contacts and can be used for any contact approach, analytical, semi-analytical and numerical.

The circular plate, because of its geometry enters into finite bodies category.

Several numerical methods were used to investigate plate problems. The obtained results were in good agreement with theoretical ones.

For experimental analysis, an accurate method, with great repeatability, without wear it’s that of laser profilometry. With this method not only the deformed surface can be measured but also the pressure in contact by reflectivity.

A test rig was build by Glovnea to analyze contact between a sapphire flat and different shaped punches via profilometry method. Ovcharenko build a experimental set-up to analyze direct, *in situ*, by optical technique, the contact between a sapphire flat and a sphere during loading, unloading and cyclic loading-unloading.. In both cases the obtained results were in good agreement with theoretical ones.

Future work in the field of elastic contact punch-circular plate

Should be carried out on:

1. Combining previously obtained results for finite dimension bodies in contact and applying them for clamped and simply supported circular plates.
2. Developing a numerical program for contact analysis and stress state evaluation for both simply supported and clamped circular plates.
3. Modifying an existent experimental set-up or building a new one, in order to permit contact analysis via laser profilometry for both simply supported and clamped sapphire circular plates.
4. Contact analysis between punches with different shapes and plates with different supports watching the evolution of contact area with load.

References


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